Introduction to Mathematical Quantum Theory

Solution to the Exercises

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Anyone who is not shocked by quantum theory has not understood it.
Niels Bohr

Exercise 1

Let $\mathcal{H} = L^2(\mathbb{R}^2)$. Let $\widetilde{H}$ be defined as

$$\widetilde{H} := -\frac{1}{2}(\Delta_x + \Delta_y) + \frac{1}{2}(x^2 + y^2) - \lambda xy \tag{1}$$

with $D\left(\widetilde{H}\right) = C_c^\infty(\mathbb{R}^2)$.

Prove that if $\lambda \in (-1, 1)$ then $\widetilde{H}$ is essentially self adjoint and study the spectrum of the closure of $\widetilde{H}$.

Hint: Prove that, with the right change of variables $(x, y) \to (w, z)$, $\widetilde{H} = H_w + H_z$ with H_w only depending on w and H_z only depending on z.

Proof. Consider the change of variables given as $z := x + y,\ w = x - y$. If we define $\phi(z, w) := \psi\left(\frac{z+w}{2}, \frac{z-w}{2}\right)$ we get that

$$\begin{aligned}\Delta_x\psi(x, y) &= \Delta_x[\phi(x + y, x - y)] = \partial_x[(\partial_z\phi)(x + y, x - y) + (\partial_w\phi)(x + y, x - y)]\\ &= (\Delta_z\phi)(x + y, x - y) + 2(\partial_z\partial_w\phi)(x + y, x - y) + (\Delta_w\phi)(x + y, x - y)\end{aligned}$$

and analogously

$$\Delta_y\psi(x, y) = (\Delta_w\phi)(x + y, x - y) - 2(\partial_z\partial_w\phi)(x + y, x - y) + (\Delta_w\phi)(x + y, x - y).$$

Now, it is easy to check that

$$\begin{aligned}x^2 + y^2 &= \frac{z^2 + w^2}{2}\\ xy &= \frac{z^2 - y^2}{4}.\end{aligned}$$

As a consequence, we get that $\psi \in C_c^\infty(\mathbb{R}^2)$ if and only if $\phi \in C_c^\infty(\mathbb{R}^2)$ and moreover

$$\widetilde{H}\psi(x, y) = \left[-(\Delta_z\phi) + \frac{1-\lambda}{4}z^2\right]\phi(z, w) + \left[-(\Delta_w\phi) + \frac{1+\lambda}{4}w^2\right]\phi(z, w).$$

If we denote now $H_\omega := -\Delta + \omega^2\xi^2$ as the harmonic oscillator in one dimension with variable ξ, we know that H_ω is self adjoint with domain $D_\omega := \left\{f \in L^2(\mathbb{R}) \mid \xi^2 f,\ k^2\widehat{f} \in L^2(\mathbb{R})\right\}$.

As a consequence, the operator defined as $H_{\sqrt{1-\lambda/2}} \otimes \mathrm{id} + \mathrm{id} \otimes H_{\sqrt{1+\lambda/2}}$ is self-adjoint with domain[1] $D_{\sqrt{1-\lambda/2}} \otimes D_{\sqrt{1+\lambda/2}}$. Given that this operator corresponds to the closure of $\widetilde{H}$, we get that $\widetilde{H}$ is essentially self-adjoint.

Now, in the exercise session we saw that

$$\sigma\left(H_{\sqrt{1-\lambda/2}} \otimes \mathrm{id} + \mathrm{id} \otimes H_{\sqrt{1+\lambda/2}}\right) = \overline{\sigma\left(H_{\sqrt{1-\lambda/2}}\right) + \sigma\left(H_{\sqrt{1+\lambda/2}}\right)},$$

and in class we saw that[2] $\sigma(H_\omega) = \omega + 2\omega\mathbb{N}$, therefore we can conclude that

$$\sigma\left(\widetilde{H}^{\mathrm{cl}}\right) = \left\{\frac{\sqrt{1+\lambda}+\sqrt{1-\lambda}}{2} + \sqrt{1+\lambda}n + \sqrt{1-\lambda}m \mid n, m \in \mathbb{N}\right\}.$$

□

Exercise 2

Let A be a normal matrix (meaning that $AA^* = A^*A$) and p a polynomial in two variables. Show by example that an eigenvector for $p(A, A^*)$ is not necessarily an eigenvector for A.

Remark: Even if eigenvectors of $p(A, A^*)$ do not correspond to eigenvectors of A, the spectrum does, in the sense that

$$\sigma(p(A, A^*)) = \{p(\lambda, \lambda^*) \mid \lambda \in \sigma(A)\}. \tag{2}$$

Proof. Consider the matrix A defined as

$$A := \begin{pmatrix} 0 & 1+i \\ -1-i & 0 \end{pmatrix}.$$

We can compute explicitly the adjoint matrix A^* and we get

$$A^* = \begin{pmatrix} 0 & -1+i \\ 1-i & 0 \end{pmatrix} = iA.$$

As a consequence we get $[A, A^*] = i[A, A] = 0$ and A is therefore normal. Let now $p(x, y) = xy$. We get that $p(A, A^*) = iA^2$. Now, another explicit computation gives us that

$$p(A, A^*) = iA^2 = i\begin{pmatrix} -2i & 0 \\ 0 & -2i \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} = 2\,\mathrm{id}.$$

[1] Recall that given two vector subspaces $V_1,\ V_2$ of $L^2(\mathbb{R})$, we have that the space $V_1 \otimes V_2$ is defined as the closure in $L^2\left(\mathbb{R}^2, dxdy\right)$ of all possible linear combinations of product of one element of V_1 with one element of V_2, i.e.

$$V_1 \otimes V_2 = \overline{\left\{\sum_{j=1}^{N} v_j^1(x)\, v_j^2(y) \mid v_j^1 \in V_1,\ v_j^2 \in V_2,\ \forall j = 1, \ldots, N\right\}}.$$

[2] Recall that $0 \in \mathbb{N}$.

Now, given that $p(A, A^*) = 2\,\mathrm{id}$, both $e_1 := (1,0)^{\mathrm{T}}$ and $e_2 := (0.1)^{\mathrm{T}}$ are eigenvector, but $Ae_1 = -(1+i)\,e_2$ and $Ae_2 = (1+i)\,e_1$, which shows that this is in fact a counter example.

□

Exercise 3

Let $I := [0,1]$ and consider $\mathcal{H} = L^2(I)$. Define the operator $H := -\Delta$ with domain[3] $D(H) := H^2(I) \cap C^1_{\mathrm{per}}(I)$. Prove that H is self-adjoint and exhibit its spectral measure explicitly.

Proof. Given that H is symmetric, we get that $D(H) \subseteq D(H^*)$. Let now $\psi \in D(H^*)$. Recall that if

$$\mathcal{F}\psi(k) = \hat{f}(k) = \int_I e^{-2\pi ikx} f(x)\,dx$$

is the Fourier series associated to ψ. We then have that

$$\begin{aligned} \mathcal{F}(\partial_x\psi)(k) &= 2\pi ik\hat{\psi}(k) \\ \mathcal{F}(\Delta_x\psi)(k), &= -(2\pi k)^2\,\hat{\psi}(k). \end{aligned}$$

Also, we mentioned before the fact that the Fourier series acts as a unitary operator. Consider now the state ψ_Λ defined as

$$\psi_\Lambda(x) = \sum_{k=-\Lambda}^{\Lambda} (2\pi k)^2\,\hat{\psi}(k)\,e^{2\pi ikx}.$$

Clearly $\hat{\psi}_\Lambda \in D(H)$. From the definition of Fourier transform we get that $\hat{\psi}_\Lambda(k) = (2\pi k)^2\,\hat{\psi}(k)$ for any $|k| \leqslant \Lambda$ and $\hat{\psi}_\Lambda(k) = 0$ otherwise. Moreover we have

$$\|\psi_\Lambda\|^2_{L^2(I)} = \sum_{k=-\Lambda}^{\Lambda} (2\pi k)^4\left|\hat{\psi}(k)\right|^2.$$

Now, we know that $|\langle H\psi_\Lambda, \psi\rangle| \leqslant C\,\|\psi_\Lambda\|_{L^2(I)}$, therefore

$$\begin{aligned} C\,\|\psi_\Lambda\|_{L^2(I)} \geqslant |\langle H\psi_\Lambda, \psi\rangle| &= \left|\sum_{k=-\Lambda}^{\Lambda} (2\pi k)^2\,\overline{\hat{\psi}_\Lambda(k)}\hat{\psi}(k)\right| \\ &= \left|\sum_{k=-\Lambda}^{\Lambda} (2\pi k)^4\left|\hat{\psi}(k)\right|^2\right| = \|\psi_\Lambda\|_{L^2(I)}. \end{aligned}$$

[3] This definition makes sense, because we know that for any function $\psi \in H^2(I)$ we have that there is a function $\tilde{\psi} \in C^1(I)$ that coincides almost everywhere with ψ. The definition of the domain is then the set of functions $\psi \in H^2(I)$ such that the function $\tilde{\psi}$ is periodic with derivative which is periodic.

This implies that $\sup_{\Lambda\in\mathbb{N}}\|\psi_\Lambda\|_{L^2(I)} \leqslant C$, and therefore $\psi \in H^2(I)$. Now, consider $\varphi \in D(H)$. Integrating by part we get

$$\begin{aligned}\langle\varphi, H^*\psi\rangle = \langle H\varphi,\psi\rangle &= \int_I \overline{-\partial_x^2\varphi(x)}\psi(x)\,dx\\ &= -\partial_x\varphi(1)\left[\psi(1)-\psi(0)\right] + \int_I \overline{\partial_x\varphi(x)}\partial_x\psi(x)\,dx\\ &= -\partial_x\varphi(1)\left[\psi(1)-\psi(0)\right] + \varphi(1)\left[\partial_x\psi(1)-\partial_x\psi(0)\right]\\ &\quad + \int_I \overline{\varphi(x)}\left(-\partial_x^2\psi(x)\right)dx.\end{aligned}$$

Considering functions such that $\varphi(0)=\partial_x\varphi(0)=0$, we get that $H^*\psi(x) = -\partial_x^2\psi(x)$. As a consequence we get that for any function $\psi \in D(H)$

$$-\partial_x\varphi(1)\left[\psi(1)-\psi(0)\right] + \varphi(1)\left[\partial_x\psi(1)-\partial_x\psi(0)\right] = 0.$$

as a consequence, $\psi(1)=\psi(0)$ and $\partial_x\psi(1)=\partial_x\psi(0)$ and $\psi \in D(H)$ and therefore H is self-adjoint.

We now get that for any $\varphi \in \mathcal{H}$ and $\psi \in D(H)$

$$\langle\varphi, H\psi\rangle = \sum_{k\in\mathbb{N}}(2\pi k)^2\,\overline{\widehat{\varphi}(k)}\widehat{\psi}(k) = \langle\varphi, \sum_{k\in\mathbb{N}}(2\pi k)^2\,\widehat{\psi}(k)\,e^{2\pi ikx}\rangle = \langle\varphi, \sum_{k\in\mathbb{N}}(2\pi k)^2\,P_k\psi\rangle,$$

where P_k is the projector along the function $e^{2\pi ikx}$. Therefore, given that $H = \sum_{k\in\mathbb{N}}(2\pi k)^2\,P_k$, the spectrum of H is given by $\sigma(H) = \left\{4\pi^2k^2 \,|\, k\in\mathbb{N}\right\}$.

We can then write H as

$$H = \sum_{\lambda\in\sigma(H)}\lambda\left[P_{\frac{\sqrt{\lambda}}{2\pi}} + P_{-\frac{\sqrt{\lambda}}{2\pi}}\right],$$

and therefore the projection-valued measure associated to H is given by

$$\mu(E) = \sum_{\lambda\in E}\lambda\left[P_{\frac{\sqrt{\lambda}}{2\pi}} + P_{-\frac{\sqrt{\lambda}}{2\pi}}\right],$$

for any E measurable subset of $\sigma(H)$.

$\square$

Exercise 4

Let $\mathcal{H}$ be an Hilbert space and $A_+,\ A_- \in \mathcal{B}(\mathcal{H})$ such that

$$\left[A_\pm, A_\pm^*\right] = \mathrm{id}, \tag{3}$$
$$\left[A_+, A_-\right] = \left[A_+, A_-^*\right] = 0. \tag{4}$$

Let moreover $\eta,\ \zeta \in \mathbb{R}$, with $\eta > \zeta \geqslant 0$. Define

$$H := \eta\left(A_+^*A_+ + A_-^*A_-\right) + \zeta\left(A_+^*A_-^* + A_+A_-\right). \tag{5}$$

Recall that if $\theta = \frac{1}{2}\operatorname{arctanh}\left(\frac{\zeta}{\eta}\right)$, $\alpha = \sqrt{\eta^2 - \zeta^2}$, $\beta = \sqrt{\eta^2 - \zeta^2} - \eta$ and C_+ and C_- are defined as

$$C_\pm := \cosh(\theta) A_\pm + \sinh(\theta) A^*_\mp \tag{6}$$

we get

$$\left[C_\pm, C^*_\pm\right] = \mathrm{id}, \tag{7}$$
$$\left[C_+, C_-\right] = \left[C_+, C^*_-\right] = 0, \tag{8}$$
$$H = \alpha\left(C^*_+ C_+ + C^*_- C_-\right) + \beta. \tag{9}$$

a Consider $X := A^*_+ A^*_- - A_+ A_-$. Prove that X is skew-adjoint, meaning that $X^* = -X$.

b For any $t \in \mathbb{R}$ consider $U(t) := e^{-tX}$. Prove that $\{U(t)\}_{t\in\mathbb{R}}$ is a strongly continuous one-parameter unitary group such that

$$U(t) A_\pm U(-t) = \cosh(t) A_\pm + \sinh(t) A^*_\mp. \tag{10}$$

Hint: Consider for any $\psi, \varphi \in \mathcal{H}$ the function $f : \mathbb{R} \to \mathbb{R}$ defined as

$$f_\pm(t) := \langle \psi, U(t) A_\pm U(-t) \varphi \rangle. \tag{11}$$

Prove that f satisfies a closed second order differential equation and deduce (10).

c Suppose that there is a complete orthonormal system $\{\varphi_n\}_{n\in\mathbb{N}}$ for $\mathcal{H}$ such that $A^*_\pm A_\pm \varphi_n = \epsilon^\pm_n \varphi_n$, with $\epsilon^\pm_n \in \mathbb{R}$. Prove that there exist a complete orthonormal system $\{\psi_n\}_{n\in\mathbb{N}}$ for $\mathcal{H}$ such that

$$H\psi_n = \left[\alpha\left(\epsilon^+_n + \epsilon^-_n\right) + \beta\right]\psi_n. \tag{12}$$

Proof. To prove **a** is enough to notice that, given that $A_\pm$ are bounded and that $A^{**}_\pm = A_\pm$, then $X^* = \left(A^*_+ A^*_- - A_+ A_-\right)^* = A_- A_+ - A^*_- A^*_+ = -X$.

To prove **b**, define $Y := iX$; then the operator Y is is firstly bounded because X is, and moreover is now self adjoint, indeed $Y^* = (iX)^* = -iX^* = iX = Y$. We can then construct via functional calculus the operator $U(t) := e^{itY} \equiv e^{-tX}$ as a strongly continuous one-parameter unitary group. For any $\psi \in \mathcal{H}$ we then have, by the Stone theorem, that

$$\lim_{h\to 0} \frac{U(t+h) - U(t)}{h}\psi = iYU(t)\psi = -XU(t)\psi.$$

Now, consider $\psi, \varphi \in \mathcal{H}$. Define then $f_\pm(t)$ as in (11), we then get that, given that $U(t)$ is a strongly continuous one-parameter unitary group, f is a differentiable function and its derivative satisfies

$$\begin{aligned}
f'_\pm(t) &= \partial_t \langle U(-t)\psi, A_\pm U(-t)\varphi\rangle \\
&= \langle XU(-t)\psi, A_\pm U(-t)\varphi\rangle + \langle U(-t)\psi, A_\pm XU(-t)\varphi\rangle \\
&= -\langle \psi, U(t) XA_\pm U(-t)\varphi\rangle + \langle \psi, U(t) A_\pm XU(-t)\varphi\rangle \\
&= -\langle \psi, U(t)\left[X, A_\pm\right] U(-t)\varphi\rangle.
\end{aligned}$$

Consider now $[X, A_+]$; we get

$$[X, A_+] = \left[A_+^* A_-^* - A_+ A_-, A_+\right] = \left[A_+^* A_-^*, A_+\right] = \left[A_+^*, A_+\right] A_-^* = -A_-^*,$$

and similarly we get

$$\begin{aligned}
[X, A_-] &= -A_+ \\
\left[X, A_+^*\right] &= -\left[X^*, A_+^*\right] = [X, A_+]^* = -A_-^{**} = -A_- \\
\left[X, A_-^*\right] &= -\left[X^*, A_-^*\right] = [X, A_-]^* = -A_+^{**} = -A_+.
\end{aligned}$$

From the fact that $[X, A_\pm]$ is bounded we also have that $f_\pm'$ is again differentiable and we get

$$\begin{aligned}
f_\pm''(t) &= \partial_t \langle U(-t)\psi, A_\mp^* U(-t)\varphi\rangle = -\langle \psi, U(t)\left[X, A_\mp^*\right] U(-t)\varphi\rangle \\
&= -\langle \psi, U(t)(-A_\pm) U(-t)\varphi\rangle = f(t).
\end{aligned}$$

As a consequence we get that f solves the following second order ordinary differential equation

$$\begin{cases}
f_\pm'' = f_\pm, \\
f_\pm(0) = \langle \psi, A_\pm \varphi\rangle, \\
f_\pm'(0) = \langle \psi, A_\mp^* \varphi\rangle.
\end{cases}$$

From the fact that $f_\pm'' = f_\pm$, we get $f_\pm(t) = f_\pm(0)\cosh(t) + f_\pm'(0)\sinh(t)$. As a consequence we get that for any $\psi,\ \varphi \in \mathcal{H}$

$$\begin{aligned}
\langle \psi, U(t) A_\pm U(-t)\varphi\rangle = f_\pm(t) &= f_\pm(0)\cosh(t) + f_\pm'(0)\sinh(t) \\
&= \langle \psi, A_\pm \varphi\rangle \cosh(t) + \langle \psi, A_\mp^* \varphi\rangle \sinh(t) \\
&= \langle \psi, \left[\cosh(t) A_\pm + \sinh(t) A_\mp^*\right]\varphi\rangle,
\end{aligned}$$

and therefore we obtain (10).

To prove **c**, consider the pair of operators $C_\pm$ defined as in 6. From point **b** if we define $U := U(\theta)$ we get that

$$\begin{aligned}
U A_\pm U^* &= \cosh(\theta) A_\pm + \sinh(\theta) A_\mp^* = C_\pm \\
U A_\pm^* U^* &= (U A_\pm U^*)^* = C_\pm^*.
\end{aligned}$$

Using we can then rewrite the Hamiltonian as

$$H = \alpha\left(C_+^* C_+ + C_-^* C_-\right) + \beta = U\left[\alpha\left(A_+^* A_+ + A_-^* A_-\right) + \beta\right] U^*.$$

Define now $\psi_n := U\varphi_n$; then on the one hand we have $\langle \psi_n, \psi_m\rangle = \langle U\varphi_n, U\varphi_m\rangle = \langle \varphi_n, U^* U\varphi_m\rangle = \langle \varphi_n, \varphi_m\rangle = \delta_{n,m}$ so $\{\psi_n\}_{n\in\mathbb{N}}$ is an orthonormal system. Given that $\{\varphi_n\}_{n\in\mathbb{N}}$ is also complete and U is bijective, we get that also $\{\psi_n\}_{n\in\mathbb{N}}$ is a complete orthonormal system. Now we then get

$$\begin{aligned}
H\psi_n &= U\left[\alpha\left(A_+^* A_+ + A_-^* A_-\right) + \beta\right] U^* (U\varphi_n) = U\left[\alpha\left(A_+^* A_+ + A_-^* A_-\right)\varphi_n\right] + \beta\psi_n \\
&= U\left[\alpha\left(\epsilon_n^+ + \epsilon_n^-\right)\varphi_n\right] + \beta\psi_n = \left[\alpha\left(\epsilon_n^+ + \epsilon_n^-\right) + \beta\right]\psi_n,
\end{aligned}$$

which concludes the proof.

□